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High mode number stability of an axisymmetric toroidal plasma

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In the investigation of stability of a plasma confined by magnetic fields some of the most important modes of oscillation are those with long wavelength parallel to the magnetic field and short wavelength perpendicular to it. However, these characteristics conflict with the requirement of periodicity in a toroidal magnetic field with shear. This conflict can be resolved by transforming the calculation to one in an infinite domain without periodicity constraints. This transformation is the starting point for a full investigation of the magnetohydrodynamic stability of an axisymmetric plasma at large toroidal wave number n. (Small values of n can be studied by direct numerical computation but this fails when n is large.) For $n \gg 1$ there are two distinct length scales in the problem and a systematic approximation is developed around an eikonal representation, formally as an expansion in $1/n$. In lowest order the oscillations of each magnetic surface are decoupled and a local eigenvalue is obtained. However, the mode structure is not fully determined in this lowest order. In higher orders a second eigenvalue equation is obtained which completes the determination of the structure of the mode and relates the local eigenvalue of the lower order theory to the true eigenvalue for the problem. This higher order theory shows that unstable modes are localized in the vicinity of the surface with the smallest local eigenvalue, that the true eigenvalue is close to the lowest local eigenvalue and that the most unstable high n modes occur for $n \to \infty$. Hence the local theory, which involves no more than the solution of an ordinary differential equation, is normally adequate for the determination of stability of any axisymmetric plasma to high mode number oscillations.

1. Introduction

It is well known that plasmas confined by magnetic fields are often unstable. In a simple idealized configuration, such as an infinite cylinder, there is a single comprehensive test for the magnetohydrodynamic (m.h.d.) stability of the plasma, given by Newcomb (1960). For any given cylindrical equilibrium profile this requires only the solution of an ordinary differential equation. However in a more realistic toroidal configuration, such as a Tokamak or Toroidal Pinch, there is no such comprehensive method for determining stability even though the system may be axisymmetric so that Fourier modes $\propto \exp(in\zeta)$ (where ζ is the angle around the symmetry axis) may be considered individually. There are certain *necessary* criteria for m.h.d. stability, such as Mercier (1960), which can be applied and there are elaborate two-dimensional numerical codes (Wesson & Sykes 1975, Todd *et al.* 1977, Berger *et al.* 1977) which can be used to test for stability. But these two-dimensional codes can describe only oscillations of small toroidal mode number n.

In this paper we develop a method for determining m.h.d. stability of toroidal axisymmetric plasmas to high n perturbations. Since this theory encompasses all high mode number perturbations it complements the two-dimensional numerical computations and so essentially completes the ideal m.h.d. stability theory of axisymmetric toroidal systems.

From studies of simple configurations we know that some of the most persistent instabilities are those which have short wavelength perpendicular to the magnetic field but long wavelength parallel to it. (These characteristics minimize the stabilizing influence of the magnetic field.) However in a toroidal magnetic field with shear (that is when the rotational transform varies from surface to surface) these characteristics conflict with the requirement that the perturbation be periodic in both toroidal and poloidal directions. The first problem in the investigation of stability of toroidal systems is therefore that of reconciling long parallel wavelength, short perpendicular wavelength and periodicity.

In § 2 we describe a general method for achieving this reconciliation, by means of a transformation from the periodic domain to an infinite domain. With this transformation the theory of high mode number oscillations is then developed in § 3. There we show that in the leading order of an expansion in $1/n$, the oscillations of each magnetic surface ψ = const. are decoupled. The lowest order theory therefore defines a local oscillation frequency $\omega^2(\psi)$ and fixes the structure of the mode along the magnetic field. However this lowest order theory does not determine the structure of the mode transverse to the magnetic surfaces. This transverse structure, and the relation of the local oscillation frequency $\omega^2(\psi)$ to the true frequency Ω^2, are determined in higher orders of the expansion.

Although the higher order theory is necessary to complete the solution, this solution itself can be expressed entirely in terms of quantities calculated from the lowest-order theory. Consequently the lowest-order theory alone is sufficient for the determination of stability. Like Newcomb's analysis this involves no more than the solution of an ordinary differential equation which can readily be solved for any given equilibrium profile.

2. Shear, periodicity and long parallel wavelength

In this section we review the problem of reconciling long-parallel and short-perpendicular wavelength with periodicity in a sheared toroidal magnetic field and show how it may be overcome.

In any axisymmetric toroidal system the magnetic field may be expressed as $\boldsymbol{B} = \nabla\psi \times \nabla\zeta + I(\psi)\nabla\zeta$, where ψ = constant defines a toroidal magnetic surface and

ζ is the angle around the axis of the torus. We can then introduce an orthogonal coordinate system (ψ, ζ, χ) (Mercier 1960) where χ is a poloidal, angle-like, coordinate (so that χ, ζ locate a point on a magnetic surface). If R is the distance from the axis of symmetry, the metric for these coordinates is $\mathrm{d}s^2 = (\mathrm{d}\psi/RB_\chi)^2 + (JB_\chi \mathrm{d}\chi)^2 + (R\,\mathrm{d}\zeta)^2$ where the volume element $\mathrm{d}\tau = J\,\mathrm{d}\psi\,\mathrm{d}\chi\,\mathrm{d}\zeta$. A field line is defined by $\psi = \text{constant}$, $\chi = \chi_0(\zeta)$ and $(\mathrm{d}\zeta/\mathrm{d}\chi_0) = IJ/R^2 \equiv \nu$ is an important parameter of the field structure. It is related to the 'toroidal safety factor' q by $q = (2\pi)^{-1}\oint \nu\,\mathrm{d}\chi$.

Now the usual representation of short wave oscillations in a slowly varying medium is in an eikonal form $\phi \propto F \exp(\mathrm{i}nS)$, where the phase varies rapidly $(n \gg 1)$ but F and S vary slowly. In the present problem the magnetic field introduces an overwhelming anisotropy and the important oscillations are those with short wavelength transverse to the field but *long* wavelength parallel to it. Since ζ is an ignorable coordinate the appropriate eikonal form for such oscillations is

$$\phi = F(\psi, \chi) \exp\left[\mathrm{i}n\left(\zeta - \int^\chi \nu\,\mathrm{d}\chi\right)\right], \tag{1}$$

where the phase varies rapidly across the magnetic field but is constant along it. In this expression the parallel wavelength of the oscillation and the effect of the slowly varying medium (which have comparable scale lengths) are both embodied in the slowly varying function $F(\psi, \chi)$.

If no other consideration intervened (1) would indeed be the appropriate representation of a perturbation with long parallel and short perpendicular wavelength. Unfortunately, when there is shear in the magnetic field it is impossible to reconcile the expression (1) with the requirement of periodicity in the poloidal angle χ for all values of ψ, without abandoning the hypothesis that $F(\psi, \chi)$ varies slowly – and with it the whole concept of an eikonal representation.

Several attempts have been made to overcome this difficulty while retaining the form (1). In a recent calculation of ballooning modes (Dobrott *et al.* 1977) the periodicity condition was replaced by the constraint that $F = 0$ at each end of the basic interval in χ. However the most unstable mode cannot be constructed in this way so that the stability of the system is overestimated (Connor, Hastie & Taylor 1978). Another approach (Rutherford *et al* 1969) is to introduce a discontinuous change in F at the ends of a basic period to compensate for the change in the exponential factor over that period. However, except when the shear is very weak, this is contrary to the requirement that F varies slowly. Yet another technique (Connor & Hastie 1975) is to introduce an arbitrary function G into the eikonal, such that $\oint(\nu + G)\,\mathrm{d}\chi = 2\pi m$, where m is integer, on all surfaces, but no satisfactory method for determining G has been given. One choice, (Coppi & Rewoldt 1975, Coppi 1977) is $G = (-\oint \nu\,\mathrm{d}\chi)\,\delta(\chi - \chi_0)$; but the discontinuity which this introduces requires that $F(\chi_0) = 0$ and this constraint again prevents construction of the most unstable modes.

It is clear that in order to obtain the correct compromise between long parallel wavelength and periodicity an alternative representation for the perturbation is needed. The construction of this representation can be described in general terms as follows. (The method is actually an extension of that used to describe the influence of shear on drift waves (Taylor 1977) and a preliminary account has been given by Connor, Hastie & Taylor (1978).)

After Fourier decomposition $\propto \exp(in\zeta)$ in the ignorable coordinate, the calculation of linear oscillations in any axisymmetric system can always be reduced to a two-dimensional eigenvalue problem

$$\mathscr{L}(\theta, x)\,\phi(\theta, x) = \lambda\phi(\theta, x), \tag{2}$$

where θ represents the poloidal angle and x the flux surface coordinate. The operator $\mathscr{L}$ is periodic in θ, $0 < \theta \leqslant 2\pi$, and ϕ must be periodic in θ and bounded in x. We now express ϕ in the form

$$\phi(\theta, x) = \sum_m \mathrm{e}^{-\mathrm{i}m\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\eta}\,\hat{\phi}(\eta, x)\,\mathrm{d}\eta, \tag{3}$$

which automatically ensures that ϕ is periodic in θ. The function $\hat{\phi}$ need not be periodic.

This transformation from ϕ to $\hat{\phi}$ can be regarded as made up of three successive steps. In the first the periodic function ϕ is represented by a Fourier sum $\Sigma a_m \exp(-im\theta)$; in the second the coefficients a_m are extended into a function $a(m)$ coinciding with a_m when m is an integer and in the third step this function $a(m)$ is itself represented by a Fourier integral.

By direct substitution of the transformation (3) into (2) it can be seen that any $\hat{\phi}(\eta, x)$ which is a solution of

$$\mathscr{L}(\eta, x)\,\hat{\phi}(\eta, x) = \lambda\hat{\phi}(\eta, x) \tag{4}$$

in the *infinite* domain $-\infty < \eta < +\infty$ will generate a periodic solution $\phi(\theta, x)$ of (2) with the same eigenvalue. In fact, all the relevant periodic solutions of (2) can be obtained from the eigenfunctions of (4). (This and other properties of the transformation are given in appendix A.)

In effect, the transformation (3) replaces the actual stability problem, with its awkward periodicity requirement, by a fictitious problem in the infinite domain with the *same* eigenvalue. The operator for the fictitious problem is identical with that in the real problem so that properties such as short perpendicular and long parallel wavelength retain their importance. The point of the transformation is that, because it does not have to be periodic, $\hat{\phi}(\eta, x)$ (unlike $\phi(\theta, x)$) *can* be represented in an eikonal form $F(\eta, x)\exp(inS)$ with the amplitude $F(\eta, x)$ slowly varying compared to the phase function. The amplitude $F(\eta, x)$ can then be calculated as an expansion in powers of $1/n$. As will be seen, in the lowest order of the expansion F satisfies an *ordinary* differential equation in the η coordinate alone and the variation of F with x is determined by higher order equations.

It is interesting that this eikonal form for $\hat{\phi}$ is essentially the 'quasi-mode' introduced intuitively by Roberts & Taylor (1965); it is a perturbation in the form of a 'twisted slice' which is everywhere almost parallel to the magnetic field. However, here $\hat{\phi}$ is not the actual plasma perturbation. The real, periodic, perturbation $\phi(\theta, x)$ which can be constructed from it will resemble a superposition of quasi-modes (appendix A). It is this superposition of quasi-modes which is the sought for representation of oscillations with short perpendicular and long parallel wavelength in a torus.

In order to complete the eigenvalue problem in the infinite domain one needs the appropriate boundary conditions as $|\eta| \to \infty$. These follow from the requirement that $\hat{\phi}(\eta, x)$ must generate a physically acceptable $\phi(\theta, x)$. In particular, as $|\eta| \to \infty$, $\hat{\phi}$ must be such that the integration in (3) will converge and in many cases this alone is sufficient to distinguish the acceptable from the non-acceptable solutions of (4). A particularly interesting example where a more subtle test of acceptability is required is described in § 4.

3. Magnetohydrodynamic stability with $n \gg 1$

In this section we investigate the m.h.d. stability of an axisymmetric toroidal plasma to high mode number perturbations starting from the ideal m.h.d. energy principle (Bernstein *et al.* 1958). According to this the change in potential energy due to a displacement ξ is given by the functional

$$\delta W = \tfrac{1}{2}\int \mathrm{d}\tau[Q^2 - J\cdot Q \times \xi + (\xi\cdot\nabla p)(\nabla\cdot\xi) + \gamma p(\nabla\cdot\xi)^2], \tag{5}$$

where $Q = \nabla \times (\xi \times B)$ is the perturbation of the magnetic field. The given equilibrium is specified by the pressure p and the current density J, and γ is the specific heat ratio. The sign of the minimum value of $\delta W(\xi, \xi)$ with respect to ξ determines the stability of the equilibrium.

We introduce an individual Fourier mode $\xi = \xi(\psi, \chi)\exp(in\zeta)$. Then by carrying out the minimization of δW with respect to $\xi_\parallel$ (the component of ξ parallel to B) and with respect to ξ_s (the component of $\xi_\perp$ lying in the magnetic surface) the potential energy can be reduced to a quadratic form in ξ_ψ (the component of ξ normal to the magnetic surface) alone. From this one then constructs an Euler equation for the minimizing ξ_ψ.

To describe this minimization in detail it is convenient to introduce

$$X = RB_\chi \xi_\psi, \quad U = \left(\frac{\xi_\zeta}{R} - \frac{I}{R^2 B_\chi}\xi_\chi\right), \quad Z = \frac{\xi_\chi}{B_\chi}, \tag{6}$$

so that U is proportional to the displacement ξ_s. In terms of these quantities δW takes the form

$$\delta W = \frac{1}{2}\int J\,\mathrm{d}\psi\,\mathrm{d}\zeta\,\mathrm{d}\chi\left\{\frac{B^2}{R^2B_\chi^2}|k_\parallel X|^2 + \frac{R^2}{J^2}\left|\frac{\partial U}{\partial \chi} - I\frac{\partial}{\partial\psi}\left(\frac{JX}{R^2}\right)\right|^2\right.$$
$$\left. + B_\chi^2\left|inU + \frac{\partial X}{\partial \psi} + \frac{j_\zeta}{RB_\chi^2}X\right|^2 - 2K|X|^2 + \gamma p\left|\frac{1}{J}\frac{\partial}{\partial\psi}(JX) + inU + iBk_\parallel Z\right|^2\right\}, \tag{7}$$

where we have made use of the equilibrium relations for the toroidal current j_ζ and pressure gradient $p'(\psi)$ (prime denotes a ψ derivative):

$$\frac{j_\zeta}{R} = p' + \frac{II'}{R^2}, \quad j_\zeta = -\frac{R}{J}\frac{\partial}{\partial\psi}(JB_\chi^2). \tag{8}$$

The coefficient K is
$$K = \frac{II'}{R^2}\frac{\partial}{\partial\psi}(\ln R) - \frac{j_\zeta}{R}\frac{\partial}{\partial\psi}\ln(JB_\chi), \tag{9}$$

and $k_\parallel$ represents the 'parallel gradient' operator

$$\mathrm{i}k_\parallel \equiv \frac{1}{JB}\left(\frac{\partial}{\partial\chi} + \mathrm{i}n\nu\right). \tag{10}$$

The first step in minimization of δW is to select Z so that the last term in δW vanishes. This is always possible (unless the shear vanishes) and corresponds to making $\nabla\cdot\boldsymbol{\xi} = 0$.

For the next step, minimization with respect to U, we first observe that when $n \to \infty$, δW will be positive and large unless $k_\parallel X$ and $k_\parallel U$ are $O(1)$ in this limit. (This represents the predominance of the parallel gradient operator which requires that the least stable modes have long parallel wavelength.) This feature of $k_\parallel U$ allows us to carry out a systematic minimization with respect to U as an expansion in $1/n$. We replace $\partial U/\partial\chi$ in (7) by iteration of the relation

$$\partial U/\partial\chi = -\mathrm{i}n\nu U + \mathrm{i}JBk_\parallel U, \tag{11}$$

and U can then be determined in each order of $1/n$ by an algebraic minimization of δW. Correct to $O(1/n)$ the minimizing U is given by

$$inU + \frac{\partial X}{\partial\psi} + X\left(\frac{p'}{B^2} + \frac{\nu'}{\nu}\frac{I^2}{R^2B^2}\right) + \frac{I^2}{\nu R^2B^2}JBk_\parallel\left(\frac{1}{n}\frac{\partial X}{\partial\psi}\right) = 0, \tag{12}$$

which represents the fact that $\nabla\cdot\boldsymbol{\xi}_\perp$ is an $O(1)$ quantity as $n\to\infty$. By using (11) and (12) we obtain δW, correct to $O(1/n)$, in terms of X alone

$$\begin{aligned}\delta W = \pi\int \mathrm{d}\psi\,\mathrm{d}\chi\Bigg\{&\frac{JB^2}{R^2B_\chi^2}|k_\parallel X|^2 + \frac{R^2B_\chi^2}{JB^2}\left|\frac{1}{n}\frac{\partial}{\partial\psi}(JBk_\parallel X)\right|^2 \\ &-\frac{2J}{B^2}p'\left[|X|^2\frac{\partial}{\partial\psi}\left(p+\frac{B^2}{2}\right) - \frac{\mathrm{i}I}{JB^2}\frac{\partial}{\partial\chi}\left(\frac{B^2}{2}\right)\frac{X^*}{n}\frac{\partial X}{\partial\psi}\right] \\ &+\frac{X^*}{n}JBk_\parallel(X\sigma') - \frac{1}{n}[P^*JBk_\parallel Q + PJBk_\parallel^* Q^*]\Bigg\},\end{aligned} \tag{13}$$

where
$$P = X\sigma - \frac{B_\chi^2}{\nu B^2}\frac{I}{n}\frac{\partial}{\partial\psi}(JBk_\parallel X), \quad Q = \frac{Xp'}{B^2} + \frac{I^2}{\nu R^2B^2}\frac{1}{n}\frac{\partial}{\partial\psi}(JBk_\parallel X),$$
$$\sigma = \frac{Ip'}{B^2} + I'.$$

This expression (13) forms the starting point for the investigation of high mode number perturbations. It must be minimized with respect to all periodic functions X subject to an appropriate normalization, which we take to be

$$\pi \int J\,\mathrm{d}\psi\,\mathrm{d}\chi \left\{\frac{|X|^2}{R^2B_\chi^2} + \left(\frac{RB_\chi}{B}\right)^2 \left|\frac{1}{n}\frac{\partial X}{\partial \psi}\right|^2\right\} = 1. \tag{14}$$

This represents the kinetic energy of the transverse motion (to leading order in $1/n$) and is convenient because it retains most of the features of the total energy normalization without affecting the minimization of δW with respect to Z and U.

The two-dimensional Euler equation for the minimizing function $X(\psi, \chi)$ is then

$$\begin{aligned} &JBk_\parallel \left\{\frac{1}{JR^2B_\chi^2}\left[1 - \left(\frac{R^2B_\chi^2}{B}\right)^2 \frac{1}{n^2}\frac{\partial^2}{\partial\psi^2}\right] JBk_\parallel X\right\} - \frac{2J}{B^2} Xp' \frac{\partial}{\partial\psi}(p + \tfrac{1}{2}B^2) \\ &\quad + \frac{\mathrm{i}}{n}\frac{\partial X}{\partial\psi}\frac{p'I}{B^4}\frac{\partial B^2}{\partial\chi} - \frac{1}{n} JBk_\parallel \left[\frac{\partial}{\partial\psi}\left(\frac{R^2B_\chi^2}{JB^2}\right)\frac{1}{n}\frac{\partial}{\partial\psi}(JBk_\parallel X)\right] \\ &\quad + \frac{1}{n} JBk_\parallel(\sigma' X) - \frac{\sigma}{n} JBk_\parallel Q - \frac{p'}{nB^2} JBk_\parallel P \\ &\quad - \frac{1}{n} JBk_\parallel \left\{\frac{1}{n}\frac{\partial}{\partial\psi}\left[\frac{IB_\chi^2}{\nu B^2} JBk_\parallel Q\right]\right\} + \frac{1}{n} JBk_\parallel \left\{\frac{1}{n}\frac{\partial}{\partial\psi}\left[\frac{I^2}{\nu R^2B^2} JBk_\parallel P\right]\right\} \\ &= \Omega^2\left[\frac{J}{R^2B_\chi^2} X - \frac{JR^2B_\chi^2}{B^2}\frac{1}{n^2}\frac{\partial^2 X}{\partial\psi^2} - \frac{1}{n}\frac{\partial}{\partial\psi}\left(\frac{JR^2B_\chi^2}{B^2}\right)\frac{1}{n}\frac{\partial X}{\partial\psi}\right]. \end{aligned} \tag{15}$$

This partial differential equation, with the periodicity condition $X(\chi + \chi_0) = X(\chi)$ where $\chi_0 = \oint \mathrm{d}\chi$, determines the stability of the system through the sign of its lowest eigenvalue Ω^2. It is an equation of the general type discussed in § 2 and hence it is amenable to the transformation described there:

$$X(\psi, \chi) = \sum_m \exp\left(-\frac{2\pi \mathrm{i} m\chi}{\chi_0}\right) \int_{-\infty}^{\infty} \mathrm{d}y \exp\left(\frac{2\pi \mathrm{i} m y}{\chi_0}\right) \hat{X}(\psi, y). \tag{16}$$

This converts equation (15) for X into an identical equation for $\hat{X}$ but with $\hat{X}$ in the infinite domain and free of periodicity requirements. Because $\hat{X}$ *is* free of periodicity constraints it can be represented in the form

$$\hat{X}(\psi, y) = F(\psi, y) \exp\left(-\mathrm{i}n \int_{y_0}^{y} \nu\,\mathrm{d}y\right). \tag{17}$$

in which all the rapid variation of $\hat{X}$ is contained in the exponential phase factor and the amplitude $F(\psi, y)$ remains a more slowly varying function as $n \to \infty$.

To demonstrate this formally, we introduce two length scales in the direction normal to the magnetic surfaces: the equilibrium scale which we continue to denote by ψ and a more rapid scale $x = n^{\frac{1}{2}}(\psi - \psi_0)$, where ψ_0 will be identified later. Then when (16) and (17) are introduced into the eigenvalue equation (15) the result can be written

$$(\mathrm{L} + \Omega^2\mathrm{M})\,F = 0, \tag{18}$$

where

$$\left.\begin{aligned} \mathrm{L} &= \mathrm{L}_0 + \frac{1}{n^{\frac{1}{2}}}\mathrm{L}_1 + \frac{1}{n}\mathrm{L}_2, \\ \mathrm{M} &= \mathrm{M}_0 + \frac{1}{n^{\frac{1}{2}}}\mathrm{M}_1 + \frac{1}{n}\mathrm{M}_2. \end{aligned}\right\} \tag{19}$$

The leading order operators L_0 and M_0 are

$$\mathrm{L}_0 F = \frac{\partial}{\partial y}\left\{\frac{1}{JR^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)^2\right]\frac{\partial F}{\partial y}\right\} + F\left\{\frac{2Jp'}{B^2}\frac{\partial}{\partial\psi}(p+\tfrac{1}{2}B^2) - \frac{Ip'}{B^4}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial B^2}{\partial y}\right\}. \tag{20a}$$

and

$$\mathrm{M}_0 F = \frac{J}{R^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)^2\right]F. \tag{20b}$$

Note that L_0 is a differential operator in the extended parallel coordinate $y(-\infty < y < \infty)$ *alone* and depends only parametrically on the coordinate ψ. If we write

$$\mathrm{L}_0 = \mathrm{L}_0\left(\frac{\partial}{\partial y}, y; \psi, y_0\right), \quad \mathrm{M}_0 = \mathrm{M}_0(y; \psi, y_0),$$

then the higher order operators, which are given in full in appendix B, can be written

$$\mathrm{L}_1 = \widehat{\mathrm{L}}_1\,\mathrm{i}\frac{\partial}{\partial x}, \quad \mathrm{M}_1 = \widehat{\mathrm{M}}_1\,\mathrm{i}\frac{\partial}{\partial x}, \tag{21}$$

with

$$\widehat{\mathrm{L}}_1 = -\frac{1}{\nu'(y_0)}\frac{\partial \mathrm{L}_0}{\partial y_0}, \quad \widehat{\mathrm{M}}_1 = -\frac{1}{\nu'(y_0)}\frac{\partial \mathrm{M}_0}{\partial y_0},$$

and

$$\mathrm{L}_2 = -\widehat{\mathrm{L}}_2\frac{\partial^2}{\partial x^2} + \widetilde{\mathrm{L}}_2, \quad \mathrm{M}_2 = -\widehat{\mathrm{M}}_2\frac{\partial^2}{\partial x^2} + \widetilde{\mathrm{M}}_2, \tag{22}$$

with

$$\widehat{\mathrm{L}}_2 = \frac{1}{2\nu'(y_0)}\frac{\partial}{\partial y_0}\left(\frac{1}{\nu'(y_0)}\frac{\partial \mathrm{L}_0}{\partial y_0}\right), \quad \widehat{\mathrm{M}}_2 = \frac{1}{2\nu'(y_0)}\frac{\partial}{\partial y_0}\left(\frac{1}{\nu'(y_0)}\frac{\partial \mathrm{M}_0}{\partial y_0}\right),$$

$\widetilde{\mathrm{L}}_2$ and $\widetilde{\mathrm{M}}_2$ also being differential operators in y alone.

We now seek a solution of (18) by an expansion in powers of $1/n^{\frac{1}{2}}$. The lowest order approximation is

$$[\mathrm{L}_0 + \omega^2(\psi, y_0)\,\mathrm{M}_0]\,F_0 = 0, \tag{23}$$

or explicitly

$$\frac{1}{J}\frac{\partial}{\partial y}\left\{\frac{1}{JR^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)^2\right]\frac{\partial F_0}{\partial y}\right\} + 2\frac{F_0\,p'}{B^2}\left[\frac{\partial}{\partial\psi}(p+\tfrac{1}{2}B^2) - \frac{I}{B^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{1}{J}\frac{\partial}{\partial y}(\tfrac{1}{2}B^2)\right] + \frac{\omega^2(\psi, y_0)}{R^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)^2\right]F_0 = 0. \tag{24}$$

Thus the lowest order approximation yields an eigenvalue problem in one dimension only. The oscillations of each surface are decoupled and each has an oscillation frequency $\omega^2(\psi, y_0)$ which depends on the flux surface ψ and the quasi-mode origin y_0. Because (24) is a differential equation in y alone the eigenfunction can be multiplied by an arbitrary function of x and is therefore of the form

$$F_0 = A(x) f_0(y; \psi, y_0), \tag{25}$$

where the variation of f_0 with ψ arises only from the parametric dependence of L_0 on the equilibrium profile.

To calculate the eigenvalue $\omega^2(\psi, y_0)$ the boundary conditions for f_0 as $|y| \to \infty$ are required. To find these we must examine the behaviour of the two solutions of (24) at large $|y|$. If $\omega^2 < 0$ one of these two solutions is exponentially growing as $|y| \to \infty$ and one is exponentially damped. Clearly the growing solution is unacceptable and the appropriate boundary condition on f_0 is therefore simply that $f_0 \to 0$ as $|y| \to \infty$. Hence the determination of unstable solutions of (24), if they exist, is straightforward: we simply solve (24) as an orthodox two point eigenvalue equation.

On the other hand, when $\omega^2 > 0$ the two solutions of (24) both behave like $(1/y) \exp(\mathrm{i}\omega y)$ as $|y| \to \infty$ and both are acceptable. Thus an acceptable solution of (24) can be constructed for *any* positive ω^2. [This is presumably related to the existence of a continuous spectrum of stable modes for a cylindrical plasma (Grad 1973).] When $\omega^2 = 0$ a more detailed investigation is necessary. This case, which is of special interest because it leads directly to a necessary criterion for the stability of all high n m.h.d. modes, is discussed in § 4.

In the lowest order calculation the 'envelope' $A(x)$, the origin y_0 of the quasi-mode and the relation of $\omega^2(\psi, y_0)$ to the global eigenvalue Ω^2 are all undetermined. To resolve this indeterminacy one must proceed to higher order in the $1/n^{\frac{1}{2}}$ expansion. The next order yields the equation

$$(\mathrm{L}_0 + \omega^2 \mathrm{M}_0) F_1 + (\mathrm{L}_1 + \omega^2 \mathrm{M}_1) F_0 = 0, \tag{26}$$

and it is clear from (21) and (25) that

$$F_1 = \mathrm{i} \frac{\mathrm{d}A}{\mathrm{d}x} f_1, \tag{27}$$

where
$$(\mathrm{L}_0 + \omega^2 \mathrm{M}_0) f_1 + (\widehat{\mathrm{L}}_1 + \omega^2 \widehat{\mathrm{M}}_1) f_0 = 0. \tag{28}$$

An integrability condition, for the existence of f_1, is obtained using the self-adjointness of the operator $(\mathrm{L}_0 + \omega^2 \mathrm{M}_0)$ and the fact that f_0 satisfies (23), (two properties which will be exploited frequently in subsequent analysis). This integrability condition is

$$\langle f_0 | \widehat{\mathrm{L}}_1 + \omega^2 \widehat{\mathrm{M}}_1 | f_0 \rangle = 0 \tag{29}$$

(with the obvious notation
$$\langle f | \mathrm{L} | g \rangle = \int_{-\infty}^{\infty} \mathrm{d}y\, f \mathrm{L} g \Bigg).$$

Differentiating (23) with respect to y_0 shows that the condition (29) is equivalent to the more useful result

$$\frac{\partial}{\partial y_0}\omega^2(\psi, y_0) = 0. \tag{30}$$

This fixes the hitherto undetermined parameter y_0: on each magnetic surface ψ it must be located at an extremum of $\omega^2(\psi, y_0)$. In most cases the location of the extrema will be obvious from the symmetry of the system and such that y_0 is independent of ψ – as we have assumed. (The general case can be incorporated by adding $in\int\nu(\mathrm{d}y_0/\mathrm{d}\psi)\,\mathrm{d}\psi$ to the phase of the eikonal (17) so that (20) remains unchanged.)

An equation for the amplitude $A(x)$ is obtained from a similar integrability condition on the next order equation for F_2,

$$(\mathrm{L}_0+\omega^2\mathrm{M}_0)\,F_2+(\mathrm{L}_1+\omega^2\mathrm{M}_1)\,F_1+(\mathrm{L}_2+\omega^2\mathrm{M}_2)\,F_0+n(\Omega^2-\omega^2)\,\mathrm{M}_0 F_0 = 0. \tag{31}$$

This integrability condition is

$$\langle f_0|\,\mathrm{L}_1+\omega^2\mathrm{M}_1\,|F_1\rangle+\langle f_0|\,\mathrm{L}_2+\omega^2\mathrm{M}_2\,|F_0\rangle+\left[n(\Omega^2-\omega_0^2)-\frac{1}{2}\frac{\partial^2\omega^2}{\partial\psi^2}x^2\right]\langle f_0|\,\mathrm{M}_0\,|F_0\rangle = 0, \tag{32}$$

where ψ_0 has now been chosen to be at a minimum of $\omega^2(\psi, y_0)$ (with y_0 determined as above) and we have expanded $\omega^2(\psi)$ about that minimum (in anticipation of the fact that the envelope $A(x)$ is localized in the neighbourhood of ψ_0). At this point we note an important property of the second order operator $\widetilde{\mathrm{L}}_2$ (see appendix B), namely

$$\langle f_0|\,\widetilde{\mathrm{L}}_2+\omega^2\widetilde{\mathrm{M}}_2\,|f_0\rangle = \frac{\mathrm{i}}{2}\frac{\partial}{\partial\psi}\langle f_0|\,\widehat{\mathrm{L}}_1+\omega^2\widehat{\mathrm{M}}_1\,|f_0\rangle-\frac{\mathrm{i}}{2}\frac{\partial\omega^2}{\partial\psi}\langle f_0|\,\widehat{\mathrm{M}}_1\,|f_0\rangle. \tag{33}$$

From this property and (29) it follows that

$$\langle f_0|\,\widetilde{\mathrm{L}}_2+\omega^2\widetilde{\mathrm{M}}_2\,|f_0\rangle = 0 \tag{34}$$

on $\psi = \psi_0$. Then, by using the properties (21) and (22) of the operators, the result (30) and the identity

$$\frac{\partial^2}{\partial y_0^2}\langle f_0|\,\mathrm{L}_0+\omega^2\mathrm{M}_0\,|f_0\rangle = 0, \tag{35}$$

equation (32) becomes

$$\frac{\partial^2\omega^2}{\partial y_0^2}\frac{\mathrm{d}^2A}{\mathrm{d}x^2}+(\nu'(y_0))^2\left[2n(\Omega^2-\omega_0^2)-\frac{\partial^2\omega^2}{\partial\psi^2}x^2\right]A = 0. \tag{36}$$

(A simple heuristic derivation of this equation is given in appendix C.) The most unstable mode (smallest Ω^2) will be found by taking y_0, which we have shown must be at an extremum, to be at a minimum of $\omega^2(\psi, y_0)$. Then $\partial^2\omega^2/\partial y_0^2 > 0$ and $A(x)$ is a Gaussian function:

$$A(x) = \exp\left\{-\tfrac{1}{2}|\nu'(y_0)|\left(\frac{\partial^2\omega^2}{\partial\psi^2}\Big/\frac{\partial^2\omega^2}{\partial y_0^2}\right)^{\frac{1}{2}}x^2\right\}. \tag{37}$$

The corresponding eigenvalue is

$$\Omega^2 = \omega_0^2+\frac{1}{2n\,|\nu'(y_0)|}\left(\frac{\partial^2\omega^2}{\partial\psi^2}\frac{\partial^2\omega^2}{\partial y_0^2}\right)^{\frac{1}{2}}. \tag{38}$$

These results show that the 'amplitude' $A(x)$ is indeed localized near $\psi = \psi_0$ and that the eigenvalue of the overall system, Ω^2, is equal to the minimum of the 'local' eigenvalue $\omega^2(\psi, y_0)$ plus a small correction of $0(1/n)$ which is itself defined in terms of $\omega^2(\psi, y_0)$. Because this correction is positive the most unstable high-n mode occurs in the limit $n \to \infty$.

Thus, although the lowest order theory alone is incomplete, all the relevant features of the higher order calculation are expressed in terms of the function $\omega^2(\psi, y_0)$ which is obtained from the lowest order calculation! In practice therefore we need calculate only the solution of the lowest order equation (24) in order to determine both stability and the structure of the unstable modes.

4. Boundary conditions when $\omega^2 = 0$ and the Mercier criterion

In this section we consider the behaviour of solutions of (24) as $|y| \to \infty$, which we noted requires special treatment when $\omega^2 = 0$.

Before investigating its behaviour at large $|y|$, we note that (24) is an Euler equation of a variational form $\delta\hat{W}(-\infty, \infty)$, where

$$\delta\hat{W}(y_1, y_2) = \int_{y_1}^{y_2} J\,\mathrm{d}y \left\{ \left(\frac{\partial f}{\partial y}\right)^2 \frac{1}{J^2R^2B_\chi^2}\left[1 + \left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y} \nu'\,\mathrm{d}y\right)^2\right] - 2f^2\frac{p'}{B^2}\left[\frac{\partial}{\partial\psi}(p + \tfrac{1}{2}B^2) - \frac{I}{B^2J}\frac{\partial}{\partial y}(\tfrac{1}{2}B^2)\int_{y_0}^{y}\nu'\,\mathrm{d}y\right]\right\}. \quad (39)$$

This is a one-dimensional energy integral for our problem and the stability of the system is determined by the sign of $\min\delta\hat{W}(-\infty, \infty)$. This form for $\delta\hat{W}$ closely resembles that studied in the analysis of the cylindrical pinch by Newcomb (1960). We can therefore use this analysis, in particular theorem 5, which states that if a solution of the Euler equation which vanishes at y_1 also vanishes at some other point of an interval (y_1, y_2) containing no singular points, then a function $f(y)$ can be constructed such that $f(y_1) = f(y_2) = 0$ and

$$\delta\hat{W}(y_1, y_2) < 0. \quad (40)$$

In the present problem there are no singular points: hence if a solution of (24) oscillates as $|y| \to \infty$ the system must be unstable.

We now return to the behaviour of solutions of (24) as $|y| \to \infty$ when $\omega^2 = 0$. It is clear that in this limit the solution depends on the 'stretched' variable $z = \int_{y_0}^{y} \nu' dy$ and on the period of the equilibrium. We therefore write the solution at large y in the form

$$f(y) \simeq z^\alpha\left\{g_0(y) + \frac{1}{z}g_1(y) + \frac{1}{z^2}g_2(y) + \dots\right\},$$

where the $g_n(y)$ have the same period as the equilibrium. Then equating powers of z we find

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{R^2B_\chi^2}{JB^2}\frac{\mathrm{d}g_0}{\mathrm{d}y}\right] = 0 \quad (41)$$

with solution $g_0 = 1$. For g_1, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{R^2B_\chi^2}{JB^2}\left(\frac{\mathrm{d}g_1}{\mathrm{d}y}+\alpha\nu'\right)+\frac{Ip'}{B^2}\right] = 0. \tag{42}$$

The first integral, $\mathrm{d}g_1/\mathrm{d}y$, contains an arbitrary constant which must be chosen so that g_1 is indeed periodic. Then

$$\frac{\mathrm{d}g_1}{\mathrm{d}y}+\alpha\nu' = \frac{B^2\nu}{B_\chi^2}\left\{\frac{\alpha\oint\nu'\,\mathrm{d}y+p'\oint\frac{\nu\,\mathrm{d}y}{B_\chi^2}}{\oint\frac{\nu B^2}{B_\chi^2}\mathrm{d}y}\right\}-\frac{\nu p'}{B_\chi^2}. \tag{43}$$

The next order in $1/z$ provides an equation for g_2. Because g_2 is periodic it may be annihilated by integration over one period in y. The resulting equation is

$$(\alpha+1)\oint\mathrm{d}y\frac{\nu' R^2B_\chi^2}{JB^2}\left(\frac{\mathrm{d}g_1}{\mathrm{d}y}+\alpha\nu'\right)+2p'\oint\mathrm{d}y\frac{J}{B^2}\frac{\partial}{\partial\psi}(p+\tfrac{1}{2}B^2)-p'I\oint\frac{\mathrm{d}y}{B^2}\frac{\mathrm{d}g_1}{\mathrm{d}y} = 0. \tag{44}$$

After substituting for $\mathrm{d}g_1/\mathrm{d}y$ this provides an indicial equation for the index α, specifying it in terms of field line averages of the equilibrium quantities. The two values of α are

$$\alpha_{1,2} = -\tfrac{1}{2}\pm(\tfrac{1}{4}-D)^{\frac{1}{2}}, \tag{45}$$

where

$$D = \frac{p'}{(2\pi q')^2 I}\left\{\oint\frac{\nu B^2}{B_\chi^2}\mathrm{d}\chi\left[p'\oint\frac{J\,\mathrm{d}\chi}{B_\chi^2}-\frac{\partial}{\partial\psi}\left(\oint J\,\mathrm{d}\chi\right)\right]+2\pi q' I\oint\frac{\nu\,\mathrm{d}\chi}{B_\chi^2}-p'I\left(\oint\frac{\nu\,\mathrm{d}\chi}{B_\chi^2}\right)^2\right\}. \tag{46}$$

The quantity D is exactly that which appears in the Mercier (1960) stability criterion when this is expressed in the form $\frac{1}{4}-D > 0$.

In the present context we see that if $D > \frac{1}{4}$ the indices α are complex. In this event both solutions of (24) are oscillatory as $|y| \to \infty$ and, by the Newcomb theorem, the system is unstable. Thus the Mercier criterion emerges naturally from our analysis as a necessary condition for the stability of all high n m.h.d. modes. Furthermore when $D > \frac{1}{4}$ an acceptable solution of (24) always exists for $\omega^2 = 0$ as it does for $\omega^2 > 0$.

When $D < \frac{1}{4}$ the indices are real and unequal. In this case both asymptotic solutions may decay ($0 < D < \frac{1}{4}$), or one may decay and the other increase ($D < 0$). In either event only the smaller asymptotic solution is acceptable because, even though it may approach zero as $|y| \to \infty$, the larger solution can be shown to lead to a divergent δW. Thus if $D < \frac{1}{4}$ and $\omega^2 = 0$ the appropriate boundary condition for eigenfunctions of (24) is that they tend to the smaller asymptotic form as $|y| \to \infty$. (An equivalent condition is that $y^{\frac{1}{2}}f(y) \to 0$ as $|y| \to \infty$.)

We can now summarize the boundary conditions for (24) as $|y| \to \infty$. When $\omega^2 > 0$ both asymptotic solutions are acceptable. When $\omega^2 < 0$ (the unstable case) only the decaying solution is acceptable and the boundary condition is $f_0 \to 0$. When $\omega^2 = 0$ two cases must be distinguished: if $D > \frac{1}{4}$ both asymptotic solutions are acceptable (but the system is unstable), while if $D < \frac{1}{4}$ only the smaller solution is acceptable and the boundary condition is $y^{\frac{1}{2}}f_0 \to 0$.

5. Conclusions

In the investigation of high mode number oscillations of an axisymmetric toroidal plasma one must reconcile long parallel and short perpendicular wavelength with periodicity in a sheared magnetic field. This problem has been overcome with the help of a transformation which converts the problem into one in an infinite domain y without periodicity constraints. Then, *and only then*, one is able to introduce an eikonal or quasi-mode form

$$\hat{X} = F(\psi, y) \exp\left(-in \int_{y_0}^{y} \nu \, dy\right),$$

in which the rapid short wavelength variation is contained in the exponential factor (through $n \gg 1$) and the amplitude varies only slowly. The quasi-mode $\hat{X}$ is not the physical perturbation; this can be constructed from $\hat{X}$ and will resemble a superposition of quasi-modes.

The existence of two distinct length scales then provides the basis for a systematic calculation of the amplitude $F(\psi, y)$ and of the frequency of oscillation as an expansion in $1/n$. In lowest order the oscillations of each surface are decoupled and the 'local' eigenvalue $\omega^2(\psi, y_0)$ is determined by an ordinary differential equation (24). In this equation the independent variable is the extended poloidal coordinate $y(-\infty < y < \infty)$; the flux surface coordinate ψ and the lower limit y_0 of the quasi-mode appear only as parameters.

The boundary conditions as $|y| \to \infty$ associated with the equation for $\omega^2(\psi, y_0)$ are obtained from the requirement that any acceptable solution in the infinite domain must generate a physically acceptable function in the periodic domain when used in the transformation (3). For $\omega^2 < 0$ (the interesting case for stability analysis) the boundary condition is simply $f_0 \to 0$ as $|y| \to \infty$. For $\omega^2 > 0$ an acceptable eigenfunction can always be found: presumably a reflection of a continuous spectrum of eigenvalues. For $\omega^2 = 0$ the analysis of the asymptotic solutions as $|y| \to \infty$ shows that the present theory encompasses earlier necessary stability criteria such as that of Mercier (1960).

Solution of the 'local' lowest order equation is straightforward, but it does not determine the structure of the mode in the radial ψ coordinate. The determination of this structure, and of the relation between the 'local' eigenvalue $\omega^2(\psi, y_0)$ and the true eigenvalue Ω^2 requires a higher order theory.

In higher orders of the $1/n$ expansion one first finds a condition determining the parameter y_0: it must be at an extremum of $\omega^2(\psi, y_0)$, in fact at a minimum. Next one finds a second eigenvalue equation, this time in the radial coordinate ψ alone, which determines the mode structure and the global eigenvalue Ω^2. The coefficients of this second equation can be expressed entirely in terms of the function $\omega^2(\psi, y_0)$ obtained from the lower order calculation. Hence, although the higher order theory is essential, because the lower order theory alone is incomplete, the solution to the higher order theory involves only quantities calculated from the lowest order equation. The salient features of this solution are: (i) an unstable mode is localized in the vicinity of that magnetic surface ψ_0 which is associated with the lowest local eigenvalue $\omega^2(\psi_0) = \omega_0^2$, (ii) the true eigenvalue Ω^2 for the system is equal to ω_0^2 plus a correction of $O(1/n)$ which is itself given in terms of $\omega^2(\psi, y_0)$, and (iii) the most unstable of the high-n modes occurs in the limit $n \to \infty$ and in this limit $\Omega^2 = \omega_0^2$.

Consequently, in practical applications, the stability of any axisymmetric plasma against high mode number perturbations can be determined from the lowest order theory alone. This involves no more than an ordinary differential equation for the eigenvalue $\omega^2(\psi, y_0)$, which can readily be solved for any given equilibrium profile.

Appendix A. Transformation of periodic modes

The representation of periodic modes of long-parallel wavelength in a sheared magnetic field was achieved by the transformation,

$$\phi(\theta, x) = \sum_m \mathrm{e}^{-\mathrm{i}m\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\eta}\, \hat{\phi}(\eta, x)\, \mathrm{d}\eta, \tag{A 1}$$

which takes the perturbation from the periodic domain $0 < \theta \leqslant 2\pi$ to the infinite domain $-\infty < \eta < \infty$. In this appendix we describe some properties of this transformation.

(i) As mentioned in the main text, the transformation can be regarded as made up of three steps. In the first, the periodic function $\phi(\theta)$ is represented by a Fourier sum. This introduces the Fourier coefficients

$$a_m = \frac{1}{2\pi} \oint \mathrm{e}^{\mathrm{i}m\theta}\, \phi(\theta)\, \mathrm{d}\theta, \tag{A 2}$$

for integer m. Then in the second step this definition is extended to all m. For any well-behaved function $\phi(\theta)$ this is equivalent to defining the function $a(s)$ to be

$$a(s) = \frac{1}{\pi} \sum_m a_m \frac{\sin[(m-s)\pi]}{(m-s)}. \tag{A 3}$$

Then $a(s)$ takes the value a_m whenever s is integer, and is continuous between the integers and is consistent with (A 2).

(ii) It is clear that if $\mathscr{L}$ is a differential operator with periodic coefficients, then $\phi(\theta)$ will be a periodic solution of

$$\mathscr{L}(\partial/\partial\theta)\, \phi(\theta) = \lambda\, \phi(\theta), \tag{A 4}$$

provided $\hat{\phi}$ is a solution of

$$\mathscr{L}(\partial/\partial\eta)\, \hat{\phi}(\eta) = \lambda\, \hat{\phi}(\eta), \tag{A 5}$$

in the infinite domain $-\infty < \eta < \infty$. For, by direct substitution, followed by partial integration (assuming $\hat{\phi}$ satisfies appropriate conditions at $\pm\infty$ to ensure convergence)

$$(\mathscr{L}(\partial/\partial\theta) - \lambda)\, \phi(\theta) = \sum_m \mathrm{e}^{-\mathrm{i}m\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\eta} (\mathscr{L}(\partial/\partial\eta) - \lambda)\, \hat{\phi}(\eta)\, \mathrm{d}\eta. \tag{A 6}$$

(iii) The demonstration of the converse result, that all periodic solutions of (A 4) can be obtained from solutions of (A 5), is somewhat more involved. If $(\mathscr{L}-\lambda)\,\phi(\theta) = 0$ then, from (A 6), by Fourier's theorem,

$$\int_{-\infty}^{\infty} e^{is\eta}(\mathscr{L}(\partial/\partial\eta)-\lambda)\,\hat{\phi}(\eta)\,d\eta = 0 \quad \text{when} \quad s = \text{integer}. \tag{A 7}$$

Consequently it must be possible to write

$$\int_{-\infty}^{\infty} e^{is\eta}(\mathscr{L}-\lambda)\,\hat{\phi}(\eta)\,d\eta = q(s)\sin\pi s, \tag{A 8}$$

where $q(s)$ is bounded when s is an integer. Then Fourier integral inversion of (A 8) shows that it can be expressed as

$$(\mathscr{L}-\lambda)\,\hat{\phi}(\eta) = Q(\eta+\pi)-Q(\eta-\pi). \tag{A 9}$$

Thus it may at first appear that the vanishing of $(\mathscr{L}-\lambda)\,\phi(\theta)$ does not necessarily imply that $(\mathscr{L}-\lambda)\,\hat{\phi}(\eta)$ is zero, but only that it satisfies (A 9). However if $\hat{\phi}$ were any particular integral of (A 9) corresponding to $Q \neq 0$ then, because $\mathscr{L}$ is periodic, this could itself be expressed in the form

$$\hat{\phi}(\eta) = R(\eta+\pi)-R(\eta-\pi) \tag{A 10}$$

and in the transformation (A 1) this would give $\phi \equiv 0$ whatever the function $R(\eta)$. Thus any non-zero periodic solution of (A 4) does indeed correspond to some solution of (A 5).

(iv) An alternative view of the structure of the solution $\phi(\theta, x)$ generated by (A 1) can be obtained as follows. We write, assuming appropriate convergence properties,

$$\phi(\theta,x) = \int_{-\infty}^{\infty} \sum_m \exp(-im(\theta-\eta))\,\hat{\phi}(\eta,x)\,d\eta. \tag{A 11}$$

Then we can regard the summation over m as the definition of an (improper) periodic function

$$\sum_m \exp(-im(\theta-\eta)) = \sum_N \delta(\theta-\eta-2\pi N), \tag{A 12}$$

and hence write (A 11) as

$$\phi(\theta,x) = \sum_N \hat{\phi}(\theta-2\pi N, x). \tag{A 13}$$

Recalling that the structure of $\hat{\phi}$ is

$$\hat{\phi}(\eta,x) = F\exp\left(-in\int^{\eta}\nu\,d\eta\right), \tag{A 14}$$

we see that ϕ is indeed an infinite sum of 'quasi-modes' as mentioned in § 2.

Appendix B. Higher order operators

The full set of operators L_i and M_i appearing in (19) are given by

$$L_0F = \frac{\partial}{\partial y}\left\{\frac{1}{JR^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,dy\right)^2\right]\frac{\partial F}{\partial y}\right\}$$
$$+F\left\{\frac{2Jp'}{B^2}\frac{\partial}{\partial\psi}(p+\tfrac{1}{2}B^2)-\frac{Ip'}{B^4}\left(\int_{y_0}^{y}\nu'\,dy\right)\frac{\partial B^2}{\partial y}\right\}, \tag{B 1}$$

$$\mathrm{M}_0 F = \frac{J}{R^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)^2\right]F, \tag{B 2}$$

$$\mathrm{L}_1 F = 2\mathrm{i}\frac{\partial}{\partial y}\left[\frac{R^2B_\chi^2}{JB^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial^2 F}{\partial y\,\partial x}\right]-\frac{\mathrm{i}Ip'}{B^4}\frac{\partial B^2}{\partial y}\frac{\partial F}{\partial x}, \tag{B 3}$$

$$\mathrm{M}_1 F = 2\mathrm{i}\frac{JR^2B_\chi^2}{B^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial F}{\partial x}, \tag{B 4}$$

$$\begin{aligned}\mathrm{L}_2 F = &-\frac{\partial}{\partial y}\left[\frac{R^2B_\chi^2}{JB^2}\frac{\partial^3 F}{\partial y\,\partial x^2}\right]+\mathrm{i}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial\psi}\left(\frac{R^2B_\chi^2}{JB^2}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial F}{\partial y}\right]+\mathrm{i}\frac{\partial}{\partial y}(\sigma' F)\\ &-\mathrm{i}\sigma\frac{\partial}{\partial y}\hat{Q}-\frac{\mathrm{i}p'}{B^2}\frac{\partial\hat{P}}{\partial y}+\mathrm{i}\frac{\partial}{\partial y}\left[\frac{IB_\chi^2}{\nu B^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial\hat{Q}}{\partial y}\right]-\mathrm{i}\frac{\partial}{\partial y}\left[\frac{I^2}{\nu R^2B^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial\hat{P}}{\partial y}\right]\\ &+2\mathrm{i}\frac{\partial}{\partial y}\left[\frac{R^2B_\chi^2}{JB^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial^2 F}{\partial\psi\,\partial y}\right]-\frac{\mathrm{i}Ip'}{B^4}\frac{\partial B^2}{\partial y}\frac{\partial F}{\partial\psi},\end{aligned} \tag{B 5}$$

$$\mathrm{M}_2 F = -\frac{JR^2B_\chi^2}{B^2}\frac{\partial^2 F}{\partial x^2}+\mathrm{i}\frac{\partial}{\partial\psi}\left[\frac{JR^2B_\chi^2}{B^2}\int_{y_0}^{y}\nu'\,\mathrm{d}y\right]F+2\mathrm{i}\frac{JR^2B_\chi^2}{B^2}\left(\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\frac{\partial F}{\partial\psi}, \tag{B 6}$$

where $\hat{P} = \sigma F+\dfrac{IB_\chi^2}{\nu B^2}\left(\displaystyle\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\dfrac{\partial F}{\partial y}$, $\quad\hat{Q} = \dfrac{p'}{B^2}F-\dfrac{I^2}{\nu R^2B^2}\left(\displaystyle\int_{y_0}^{y}\nu'\,\mathrm{d}y\right)\dfrac{\partial F}{\partial y}$.

If we write L_2 and M_2 in the form

$$\mathrm{L}_2 = -\widehat{\mathrm{L}}_2\frac{\partial^2}{\partial x^2}+\widetilde{\mathrm{L}}_2,\quad \widehat{\mathrm{M}}_2 = -\widetilde{\mathrm{M}}_2\frac{\partial^2}{\partial x^2}+\mathrm{M}_2, \tag{B 7}$$

where $\widehat{\mathrm{L}}_2 = \dfrac{1}{2\nu'(y_0)}\dfrac{\partial}{\partial y_0}\left(\dfrac{1}{\nu'(y_0)}\dfrac{\partial \mathrm{L}_0}{\partial y_0}\right)$, $\quad\widehat{\mathrm{M}}_2 = \dfrac{1}{2\nu'(y_0)}\dfrac{\partial}{\partial y_0}\left(\dfrac{1}{\nu'(y_0)}\dfrac{\partial \mathrm{M}_0}{\partial y_0}\right)$, (B 8)

then the only quantity of interest which involves $\widetilde{\mathrm{L}}_2$ and $\widetilde{\mathrm{M}}_2$ is

$$I = \langle f_0|\,\widetilde{\mathrm{L}}_2+\omega^2\widetilde{\mathrm{M}}_2\,|f_0\rangle. \tag{B 9}$$

It can be shown by integration by parts over y that the terms in this expression involving $\hat{P}$ and $\hat{Q}$ vanish and that the remaining terms can be cast into the form

$$I = \frac{\mathrm{i}}{2}\frac{\partial}{\partial\psi}\langle f_0|\,\widehat{\mathrm{L}}_1+\omega^2\widehat{\mathrm{M}}_1\,|f_0\rangle-\frac{\mathrm{i}}{2}\frac{\partial\omega^2}{\partial\psi}\langle f_0|\,\widehat{\mathrm{M}}_1\,|f_0\rangle. \tag{B 10}$$

Appendix C. Equation for mode envelope $A(x)$

It is of interest to note that, if one accepts that the 'envelope' $A(x)$ of the mode must be determined from an equation of the form

$$\frac{g_2}{n}\frac{\mathrm{d}^2A}{\mathrm{d}x^2}+\frac{g_1}{n^{\frac{1}{2}}}\frac{\mathrm{d}A}{\mathrm{d}x}+(\Omega^2-\omega^2)\,A = 0, \tag{C 1}$$

then the coefficients g_2 and g_1 can be deduced without recourse to the explicit operators L_i, M_i. For the complete perturbation $\hat{X}$ is of the form

$$\hat{X} = \hat{X}_0 + \frac{1}{n^{\frac{1}{2}}}\hat{X}_1 + \dots, \tag{C 2}$$

where
$$\hat{X}_0 = \exp\left(-\mathrm{i}n\int_{y_0}^{y} \nu \,\mathrm{d}y\right) f_0(y;\, y_0, \psi)\, A(x). \tag{C 3}$$

Under the transformation

$$y_0 \rightarrow y_0 + \frac{1}{n^{\frac{1}{2}}} y_1, \quad A(x) \rightarrow A(x) \exp\left(\mathrm{i}n^{\frac{1}{2}}\nu y_1\right), \tag{C 4}$$

the lowest order term in the expansion of $\hat{X}$ remains unchanged (any change being $O(1/n^{\frac{1}{2}})$ and part of $\hat{X}_1$). The equation for $A(x)$ must reflect this invariance. Introducing the transformation (C 4) in (C 1), and equating powers of y_1 and y_1^2 in the limit $n \rightarrow \infty$ then gives

$$\mathrm{i}g_1 = \frac{1}{\nu'(y_0)}\frac{\partial \omega^2}{\partial y_0}, \quad g_2 = \frac{1}{2(\nu'(y_0))^2}\frac{\partial^2 \omega^2}{\partial y_0^2}. \tag{C 5}$$

For the most unstable mode we must then select y_0 so that $g_1 = 0$ and the result is (36) of the main text.

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